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Zonal harmonic series expansions of Legendre functions and associated Legendre functions

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Abstract. The Legendre functions $P_\nu^m(\pm\cos\vartheta)$ of complex degree ν and integral order m may be expanded in terms of Legendre functions of integral degree and order, the latter being the zonal harmonic functions $P_n^m(\cos\vartheta)$. Two methods for improving the convergence of a standard series expansion are discussed for the cases $m=0$ and 1. The first method involves repeated application of the relations between contiguous Legendre functions and the second employs integral relations between Legendre functions of different order. The formulae derived are suitable for computation and easily programmed.

1. Introduction

The Legendre functions $P_\nu(x) \equiv P_\nu^0(x)$, associated Legendre functions $P_\nu^m(x)$ with $m=1,2,3,\ldots$ and $x=\pm\cos\vartheta$ are involved in the solution of problems concerned with wave propagation or scattering when formulated in a spherical polar coordinate system (r,ϑ,φ). These functions are singular on $x=-1$ and regular on $x=+1$. For the numerical solution of such problems a computer algorithm enabling the calculation of these functions is required. The concern of this paper is to develop formulae which enable computations to be made rapidly using microcomputers, most of which have a lower precision than a mainframe computer.

If $|\nu|$ is large and ϑ is not too near 0 or π, the Legendre functions can be computed efficiently using a well known asymptotic expansion (e.g. Erdélyi *et al* 1953, p 162). It should be noted that the asymptotic expansion requires the computation of gamma functions with complex arguments. Our concern here is with values of $|\nu|$ and ϑ for which this expansion in inappropriate.

In the context of terrestrial electromagnetic wave propagation, Jones and Kemp (1970) used elementary zonal harmonic expansion formulae to compute the electric and magnetic vectors of the wavefield. Jones (1970) made calculations of electromagnetic wavefields in which the accuracy of these zonal harmonic expansion formulae for the Legendre functions (with $m=0,1$ and $0.1<|\nu|<15$) was compared to that of the asymptotic expansion referred to above and formulae derived from this asymptotic expansion.

In relation to molecular elastic scattering calculations Connor and Mackay (1978) present data listing values of $P_\nu^0(\cos\vartheta)$ computed on a mainframe for $\nu=0.1+0.1\iota$ and $\nu=3+3\iota$, using a zonal harmonic series expansion of improved convergence. The expansion used was a corrected form of a formula originally derived with considerable ingenuity by Nickolaenko and Rabinowitz (1974). In Connor and Mackay (1979), useful graphical comparisons are made of this zonal harmonic series representation of $P_\nu^0(x)$ and a variety of asymptotic formulae.

Recently, Jones and Joyce (1989) have a given a set of formulae derived from the hypergeometric series representation of the Legendre functions. These enable $P_\nu^m(-\cos\vartheta)$ to be computed efficiently on a microcomputer for $m=0$ and $1, 0<\vartheta\leqslant\pi$ and arbitrary values of ν. Values of $P_\nu^m(-\cos\vartheta)$ are tabulated for two complex values of ν for several values of ϑ.

Here we present an alternative set of exact series representations of the Legendre functions suitable for the computation of $P_\nu^m(x)$ with $m=0$ or 1, real values of x with $-1<x\leqslant+1$ and arbitrary complex values of ν. These formulae all involve series expansions in terms of the zonal harmonic functions $P_n^m(x)$, in which n and m are integers. While our primary interest is the computation of extremely low frequency electromagnetic waves propagating over the surface of the earth, for which the real part of ν is somewhat greater than the imaginary part, the formulae developed are of general validity. The values $m=0, 1$ are of particular interest in this application because these determine the electric and magnetic components of the radiated field respectively (Jones and Joyce 1989).

The series expansion formulae are derived using well known relations presented, for example, by Erdélyi *et al* (1953). We make use of two techniques, the first being the repeated application of the relations between contiguous functions and the second being the application of integral equations.

2. Convergence of zonal harmonic expansion formulae

Initially we consider the behaviour of the zonal harmonic functions for large values of n.

Sommerfeld (1967, section 24.17) showed that for $n\to\infty$, with $n\gg m$ and $0<\vartheta<\pi$, the zonal harmonics $P_n^m(\cos\vartheta)$ have the following asymptotic representation.

$$P_n^m(\cos\vartheta)\sim n^{m-1/2}\sqrt{\frac{2}{\pi\sin\vartheta}}\cos[(n+\tfrac{1}{2})\vartheta-\tfrac{1}{4}\pi+\tfrac{1}{2}m\pi]. \tag{1}$$

Hence, for large n, if $P_n^m(\cos\vartheta)$ is viewed as a function of n, m and ϑ being fixed, this function is oscillatory with period $\Delta n=2\pi/\vartheta$ and an amplitude of order $n^{m-1/2}$. As n increases this amplitude decreases if $m\leqslant 0$ and increases if $m>0$. In practice (1) gives graphical accuracy for $n\geqslant 10$ as long as ϑ is not too near 0 or π. (1) is also valid for complex values of n, i.e. $n\to\nu$, and is, in fact, the first term of the asymptotic expansion referred to in section 1 when the gamma functions in the expansion are replaced by their asymptotic expressions.

The formulae developed in this paper include the summation,

$$\sum_n f(n,\nu)P_n^m(\cos\vartheta).$$

For large n and $n\gg|\nu|$, the terms decrease as n^{-c} where the exponent c determines the rate of convergence of the summation. The value of c depends on m (as discussed in the paragraph above) and on the form of the function $f(n,\nu)$. We shall show that it is possible to obtain expansion formulae with arbitrarily large values of c (so that the convergence rate is rapid) but at the expense of having an involved formula (so that each term takes a relatively long time to compute). In considering an optimum formula for computation it is necessary to weigh the rate of convergence against the complication of the formula.

The zonal harmonic expansion formulae developed here require the computation of $P_n^m(\cos\vartheta)$ for $m=0, 1$ and arbitrary n. These functions may readily be computed by the (numerically stable) recursion relation

$$(n-m)P_n^m(x)=(2n-1)xP_{n-1}^m(x)-(n+m-1)P_{n-2}^m(x)$$

used for increasing values of n, starting with the known elementary functions $P_0^m(x)$ and $P_1^m(x)$.

The starting point of our development is the fundamental zonal harmonic expansion (Erdélyi *et al* 1953, p 167),

$$P_\nu^m(-\cos\vartheta)=(-1)^{m+1}\frac{\sin\nu\pi}{\pi}\sum_{n=0}^{\infty}\frac{(2n+1)P_n^m(\cos\vartheta)}{n(n+1)-\nu(\nu+1)}. \tag{2}$$

Formula (2) is obtained from the published formula by the replacement $\vartheta\to\pi-\vartheta$ and by using the result $P_n^m(-x)=(-1)^{m+n}P_n^m(x)$. The formula is valid for $0<\vartheta<2\pi$ and for $m\leqslant 1$. In the reference cited the latter condition is stated as $m\leqslant 0$ but numerical tests show that (2) does converge to the correct result with $m=1$. By converting the summation over discrete values of n to an integral over continuous n, it can be shown that (2) (with $m=1$) is related to the Fresnel integrals (which tend to a finite limit for $n\to\infty$) thus demonstrating the convergence of the formula.

In considering the propagation of electromagnetic waves around the Earth it is conventional to locate the radiation source (i.e. the singularity) on $\vartheta=0$. In (2) we thus choose to represent $P_\nu^m(-\cos\vartheta)$ (rather than $P_\nu^m(+\cos\vartheta)$) because this function is singular on the ray $\vartheta=0$.

From the discussion above, for values of $n\gg|\nu|$ the terms within the summation (2) are of order $n^{m-1/2}/n=n^{m-3/2}$. Formula (2) is thus poorly convergent for $m=0, 1$. For the values of $|\nu|$ appropriate for our electromagnetic wave computations (Jones and Joyce 1989) a few hundred terms are needed to compute $P_\nu^0(x)$ to graphical accuracy and thousands of terms to compute $P_\nu^1(x)$.

Formula (2), though poorly convergent, is a much simpler formula (and hence easier to programme) than the formulas involving hypergeometric functions as used by Jones and Joyce (1989). Also, in relation to the scattering of waves by a sphere, Schumann (1952) pointed out that (2) has the physical significance of being an expansion of the wavefield in terms of its normal modes, the condition $n(n+1)=\nu(\nu+1)$ giving the (complex) mode frequencies in the electromagnetic wave problem. For these reasons we have sought to develop alternative and more convergent expansion formulae involving the zonal harmonics.

3. Formulae derived by application of the contiguous relations

The relations between contiguous Legendre functions which were used are as follows (Erdélyi *et al* 1953, p 161).

$$(2\nu+1)SP_\nu^1(-C)=\nu(\nu+1)[P_{\nu+1}^0(-C)-P_{\nu-1}^0(-C)] \tag{3}$$

and

$$(2\nu+1)SP_\nu^0(-C)=-[P_{\nu+1}^1(-C)-P_{\nu-1}^1(-C)] \tag{4}$$

in which $S=\sin\vartheta$, $C=\cos\vartheta$ and $m=0$ or 1 as indicated.

Using (2) with $m=0$, and making the substitutions $\nu \to \nu+1$ and $\nu \to \nu-1$, the right-hand side of (3) can be evaluated giving,

$$\frac{\pi}{\sin \nu\pi} P_\nu^1(-C) = \frac{2\nu(\nu+1)}{\sin\vartheta} \sum_{n=0}^{\infty} \frac{(2n+1)P_n^0(C)}{[n(n+1)-\nu(\nu-1)][n(n+1)-(\nu+1)(\nu+2)]}. \tag{5}$$

The terms in the summation in (5), with $n \gg |\nu|$, are of order n^{-c} with $c=\frac{7}{2}$. The expansion (5) is thus much more convergent than the initial expansion (2) (for which, with $m=1$, $c=\frac{1}{2}$).

If (5) is now used to obtain expansions for $P_{\nu+1}^1(-C)$ and $P_{\nu-1}^1(-C)$, a new series formula for $P_\nu^0(-C)$ can be obtained using (4). The result is,

$$\frac{\pi}{\sin \nu\pi} P_\nu^0(-C) = \frac{4}{\sin^2\vartheta} \sum_{n=0}^{\infty} N_1(2n+1)P_n^0(C)/D_1 \tag{6}$$

where

$$N_1 = n(n+1)+(\nu-1)(\nu+2)$$

and

$$D_1 = [n(n+1)-(\nu-2)(\nu-1)][n(n+1)-\nu(\nu+1)][n(n+1)-(\nu+2)(\nu+3)].$$

Equation (6) is an expansion formula for $P_\nu^0(-C)$ in which, for $n \gg |\nu|$, the terms are of order n^{-c} with convergence exponent $c=\frac{7}{2}$. In contrast, for the initial formula (2) (with $m=0$), $c=\frac{3}{2}$.

Proceeding as before, we now use (6) with (3) to obtain a further expansion for $P_\nu^1(-C)$, the result being

$$\frac{\pi}{\sin \nu\pi} P_\nu^1(-C) = -\frac{16\nu(\nu+1)}{\sin^3\vartheta} \sum_{n=0}^{\infty} N_2(2n+1)P_n^0(C)/D_2 \tag{7}$$

in which

$$N_2 = 2n(n+1)+(\nu-2)(\nu+3)$$

and

$$D_2 = [n(n+1)-(\nu-3)(\nu-2)][n(n+1)-(\nu-1)\nu]$$
$$\times [n(n+1)-(\nu+1)(\nu+2)][n(n+1)-(\nu+3)(\nu+4)].$$

The terms of (7) converge as n^{-c} with $c=\frac{11}{2}$ (if $n \gg |\nu|$).

As a final illustration of this process, using (7) and (4) the result obtained is

$$\frac{\pi}{\sin \nu\pi} P_\nu^0(-C) = -\frac{64}{\sin^4\vartheta} \sum_{n=0}^{\infty} N_3(2n+1)P_n^0(C)/D_3 \tag{8}$$

with

$$N_3 = n^2(n+1)^2+2n(n+1)(2\nu^2+2\nu-7)+(\nu-3)(\nu-1)(\nu+2)(\nu+4)$$

and

$$D_3 = [n(n+1)-(\nu-4)(\nu-3)][n(n+1)-(\nu-2)(\nu-1)][n(n+1)-\nu(\nu+1)]$$
$$\times [n(n+1)-(\nu+2)(\nu+3)][n(n+1)-(\nu+4)(\nu+5)]$$

and a convergence exponent $c=\frac{11}{2}$.

It should be noted that the formulae derived above only require the computation of $P_n^0(\cos\vartheta)$ and that each time (3) is used c increases by two but when (4) is used c remains the same. While it would be possible to continue this analysis indefinitely, the derived expansion formulae are becoming so involved, that it is likely that the time saved in computation by the large convergence exponent c is more than offset by the time needed to compute each term of the summation.

All of the expansion formulae listed above have been validated numerically using a microcomputer. It appears that the formulae having $c = \frac{7}{2}$ are optimum in terms of computational speed for the range of values of ν experienced in our electromagnetic field computations ($|\nu| \leq 15$). If the imaginary part of ν is small compared to the real part, the main contribution to the summation occurs in the vicinity of $n = \operatorname{Re}\nu$ so rather more than $\operatorname{Re}\nu$ terms have to be computed whatever the convergence of the formula. Also, for small values of ϑ, numerical rounding errors lead to a loss of precision which is increasingly severe as c increases.

Formula (8) is less involved than that of Nickolaenko and Rabinowitz (1974) which has the same convergence exponent. However, the latter formula includes a partial extraction of the singularity on $\vartheta = 0$ and is advantageous for computational purposes for small values of ϑ. We develop a set of formulae in which the source singularity is extracted below.

4. Formulae obtained from the expansion of $P_\nu^{-1}(-\cos\vartheta)$

For negative integer values of m the convergence of the series expansion (2) increases as $|m|$ increases in accordance with the discussion in section 2. It should be noted that convergent series expansions for positive integer m can be obtained easily from the negative-m formulae by applying the result (Erdélyi *et al* 1953, p 144):

$$P_\nu^{-m}(x) = (-1)^m \Gamma(\nu+1-m) P_\nu^m(x)/\Gamma(\nu+1+m) \tag{9}$$

which is valid for $m = 1, 2, 3\ldots$ and both integer and complex ν. In (9) symbol Γ indicates the gamma function.

Using (9) with $m = 1$ and the result $\Gamma(\nu+2) = \nu(\nu+1)\Gamma(\nu)$ (also valid for $\nu \to n$) we get

$$P_\nu^{-1}(C) = -P_\nu^1(C)/[\nu(\nu+1)] \qquad \text{and} \qquad P_n^{-1}(C) = -P_n^1(C)/[n(n+1)]. \tag{10}$$

So from (2) (with $m = -1$), using (10) we obtain an alternative zonal harmonic expansion formula for $P_\nu^1(-C)$. The term $n = 0$ is evaluated using the result $P_0^{-1}(-C) = \cot(\frac{1}{2}\vartheta)$. The resulting expansion is then

$$\frac{\pi}{\sin\nu\pi} P_\nu^1(-C) = -\cot(\tfrac{1}{2}\vartheta) + \nu(\nu+1) \sum_{n=1}^{\infty} \frac{(2n+1)P_n^1(C)}{n(n+1)[n(n+1) - \nu(\nu+1)]}. \tag{11}$$

The terms in the summation (11), with $n \gg |\nu|$, are now of order n^{-c} with $c = \frac{5}{2}$. Equation (11) is thus significantly more convergent than the expansion (2), used with $m = 1$, for which $c = \frac{1}{2}$. The improvement is a result of the extraction of the singularity (the 'source' singularity) on the ray $\vartheta = 0$. This singularity is embodied in the term $\cot(\frac{1}{2}\vartheta)$ which tends to $2/\vartheta$ as $\vartheta \to 0$. Because $P_\nu^1(-\cos\vartheta) \to -2\sin\nu\pi/(\pi\vartheta)$ as $\vartheta \to 0$ it can be seen that the term $\cot(\frac{1}{2}\vartheta)$ correctly expresses this singularity.

Starting with (11), the technique described in section 3 can be used to generate a second set of more convergent formulae for $P_\nu^m(-C)$ with $m = 0, 1$. For example, the first expansion generated is given by (5) with the replacement of $P_\nu^1(-C)$ by $P_\nu^0(-C)$, $P_n^0(C)$ by $P_n^1(C)$ and omission of the factor $\nu(\nu+1)$. Further formulae will not be presented here but it is clear that the convergence exponents (c) of the set generated from (11) are $\frac{5}{2}, \frac{9}{2}, \frac{13}{2}, \ldots$.

As an alternative to the technique of section 3, it is possible to generate more compact expansions by integration. This is considered in the following section.

5. Formulae obtained by integration

We start with (11) and transform this to a series expansion for $P_\nu^0(-C)$ by integration. Because $P_\nu^1(-C) = -(\mathrm{d}/\mathrm{d}\vartheta)P_\nu^0(-C)$ and $P_n^1(C) = (\mathrm{d}/\mathrm{d}\vartheta)P_n^0(C)$, it follows that

$$P_\nu^0(-C) = -\int P_\nu^1(-C)\,\mathrm{d}\vartheta + K_\nu \qquad \text{and} \qquad P_n^0(C) = \int P_n^1(C)\,\mathrm{d}\vartheta + K_n$$

in which K_ν and K_n are the integration constants.

Integrating (11) a new series expansion of $P_\nu^0(-C)$ is obtained as follows:

$$P_\nu^0(-C) = \frac{\sin\nu\pi}{\pi}\left(\ln[\sin^2(\tfrac{1}{2}\vartheta)] - \nu(\nu+1)\sum_{n=1}^{\infty}\frac{(2n+1)P_n^0(C)}{n(n+1)[n(n+1)-\nu(\nu+1)]} + k_\nu\right) \ldots \tag{12}$$

In (12) the constant k_ν subsumes all the constant terms involving K_ν and the set K_n. These do not need to be considered explicitly.

To evaluate k_ν we put $\vartheta = \pi$ (i.e. $C = -1$) in (12), noting that $P_\nu^0(1) = 1$ and $P_n^0(-1) = (-1)^n$. Therefore

$$k_\nu = \frac{\pi}{\sin\nu\pi} + \nu(\nu+1)\sum_{n=1}^{\infty}\frac{(-1)^n(2n+1)}{n(n+1)[n(n+1)-\nu(\nu+1)]}. \tag{13}$$

The summation in (13) is computed by considering (2) with $m = 0$ and $\vartheta = \pi$. With some trivial manipulation, it is then easily seen that

$$\frac{\pi}{\sin\nu\pi} = \frac{1}{\nu(\nu+1)} - \sum_{n=1}^{\infty}\frac{(-1)^n(2n+1)}{n(n+1)-\nu(\nu+1)}.$$

Substituting this value of $\pi/\sin\nu\pi$ into (13) we find,

$$k_\nu = \frac{1}{\nu(\nu+1)} - \sum_{n=1}^{\infty}\frac{(-1)^n(2n+1)}{n(n+1)}$$

in which the summation is evaluated as -1 by writing out the terms of the series. Hence $k_\nu = 1 + 1/[\nu(\nu+1)]$ and (12) becomes

$$P_\nu^0(-\cos\vartheta) = \frac{\sin\nu\pi}{\pi}\left(\ln[\sin^2(\tfrac{1}{2}\vartheta)] + 1 + \frac{1}{\nu(\nu+1)} - \nu(\nu+1)\sum_{n=1}^{\infty}\frac{(2n+1)P_n^0(\cos\vartheta)}{n(n+1)[n(n+1)-\nu(\nu+1)]}\right). \tag{14}$$

For $n \gg |\nu|$, the terms in the summation in (14) are of order n^{-c} with $c = \frac{7}{2}$. It is noted that (14) has the same convergence exponent c as (6) but that the terms of (14) are less complicated than those in (6) (and hence can be computed more rapidly).

As in the discussion following (11), the simpler form of the terms in the summation in (14) is a result of the fact that the singularity on the ray $\vartheta = 0$ has been removed from the expansion. The singularity here is in the term involving $\ln[\sin^2(\frac{1}{2}\vartheta)]$. As $\vartheta \to 0$, it is well known that $P_\nu^0(-\cos\vartheta) \to (\sin\nu\pi/\pi)\ln[\sin^2(\frac{1}{2}\vartheta)]$ which is exactly the first term on the right-hand side of (14).

We now use the result (14) to obtain a new formula for $P_\nu^1(-C)$ by a further stage of integration. The basis of this next step is the result (Erdélyi *et al* 1953, p 149):

$$P_\nu^{-1}(x) = -(1-x^2)^{-1/2}\int_1^x P_\nu^0(x')\,\mathrm{d}x' \tag{15}$$

with $x \equiv -C \equiv -\cos\vartheta$, $(1-x^2) \equiv \sin^2\vartheta$ and $\mathrm{d}x \equiv \sin\vartheta\,\mathrm{d}\vartheta$. In this case there is no constant of integration to be evaluated.

The integral of the left-hand side of (14) is $\sin\vartheta\, P_\nu^1(-C)/[\nu(\nu+1)]$ and the integral of $P_n^0(C)$ on the right-hand side of (14) is $-\sin\vartheta\, P_n^1(C)/[n(n+1)]$, where (10) has been used to transform the negative-order function in (15) to positive order. After some simplification of the trigonometric terms the final result obtained is

$$P_\nu^1(-\cos\vartheta) = \frac{\sin\nu\pi}{\pi}\left(\nu(\nu+1)\tan(\tfrac{1}{2}\vartheta)\ln[\sin^2(\tfrac{1}{2}\vartheta)] - \cot(\tfrac{1}{2}\vartheta)\right.$$
$$\left. + [\nu(\nu+1)]^2 \sum_{n=1}^{\infty} \frac{(2n+1)P_n^1(\cos\vartheta)}{[n(n+1)]^2[n(n+1)-\nu(\nu+1)]}\right). \tag{16}$$

In (16) it is noted that the first term in parentheses tends to 0 as $\vartheta \to 0$, so that the singularity is the same as in (11). For $n \gg |\nu|$, the terms of the summation in (16) are of order n^{-c}, with $c = \frac{9}{2}$. Equation (16) is more convergent than the '$\frac{7}{2}$ formula' (5) (and of a similar complexity) and it is much simpler than the '$\frac{11}{2}$ formula' (7). Also the fact that the singularity on $\vartheta = 0$ is explicit in (11), (14) and (16) makes these formulae physically more satisfying than the formulae of section 3.

Because the first term in parentheses in (16) does not have a closed-form integral we cannot make further progress with this technique without introducing a second-series summation.

6. Numerical validation of the derived formulae

All of the formulae derived in this paper have been checked by making numerical computations on a 'RM Nimbus' microcomputer using a FORTRAN compiler. A set of data for comparison were calculated using the formulae involving the hypergeometric function ${}_2F_1$ as presented by Jones and Joyce (1989). The values of $|\nu|$ used in this validation were those appropriate to our electromagnetic wave propagation studies where the principal interest is in the ranges $0 < \vartheta \leq \pi$ and $0.1 < |\nu| < 15$, with a value for the imaginary part of ν which is small compared with the real part.

Because, as is evident from (1), the convergence of all the summations is oscillatory, care is needed in choosing a suitable value of n, say $n = N$, to terminate the computation. It is not generally true that increasing an arbitrarily chosen value of N will improve the accuracy of the result. N should be chosen so that the *magnitude* of the cosine function in (1) is unity, or as near unity as possible. For such values of N the sum is very close to the value obtained with $N \to \infty$, provided that $N \gg |\nu|$.

Some illustrative numerical data are presented in figures (1) to (4). These show the magnitude (or modulus) of the sum to $n = N$ terms of each of the formulae, for one

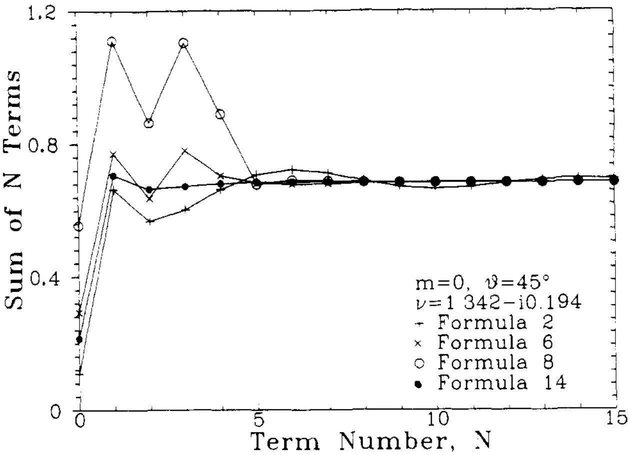

Figure 1. Sum to N terms of the formulae for $|P_\nu(\cos\vartheta)|$ with $\vartheta = 45°$ and $\nu = 1.342 - i0.194$. Data points are linked to guide the eye. ($|P_\nu| = 0.68414$ for these values of ϑ and ν, see Jones and Joyce 1989.)

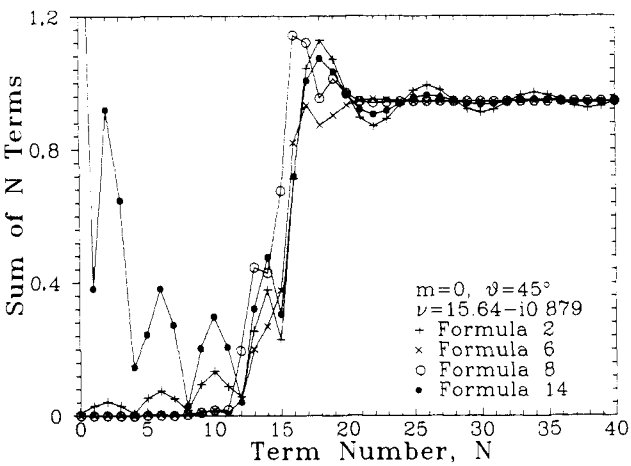

Figure 2. Sum to N terms of the formulae for $|P_\nu(\cos\vartheta)|$ with $\vartheta = 45°$ and $\nu = 15.64 - i0.879$. ($|P_\nu| = 0.94472$.)

value of ϑ, and two particular values of ν (one small and the other relatively large) of significance in our electromagnetic wavefield computations (see Jones and Joyce 1989). The differing degree of convergence of the formulae is evident, as is the cosine dependence on n predicted by Sommerfeld's formula (1), and the major contribution made by terms in the vicinity of $n = |\nu|$.

Most of our computations have been made using (14) and (16) for which, typically, 10 to 20 terms of the summations are required to compute the Legendre functions to a precision of 1 part in 200 (i.e. 'graphical accuracy'). It has been found that our computer algorithms for implementing (14) and (16) enable such calculations to be made about twice as quickly as for the much more involved algorithms which use the hypergeometric function expansions. It should be admitted, however, that the latter compute Legendre functions to the maximum precision possible on a 24 bit Mantissa

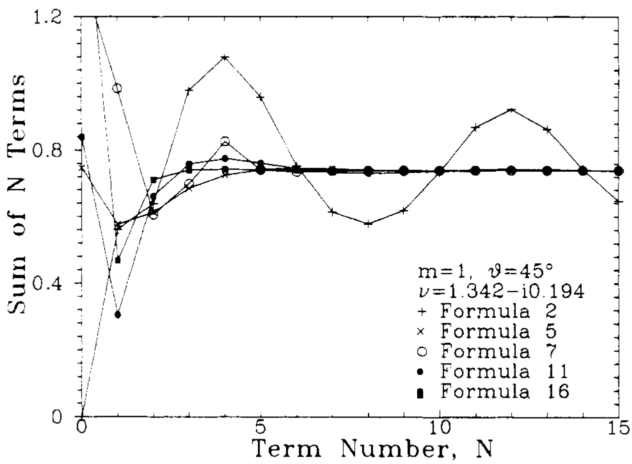

Figure 3. Sum to N terms of the formulae for $|P_\nu^1(\cos\vartheta)|$ with $\vartheta = 45°$ and $\nu = 1.342 - i0.194$. ($|P_\nu^1| = 0.73957$.)

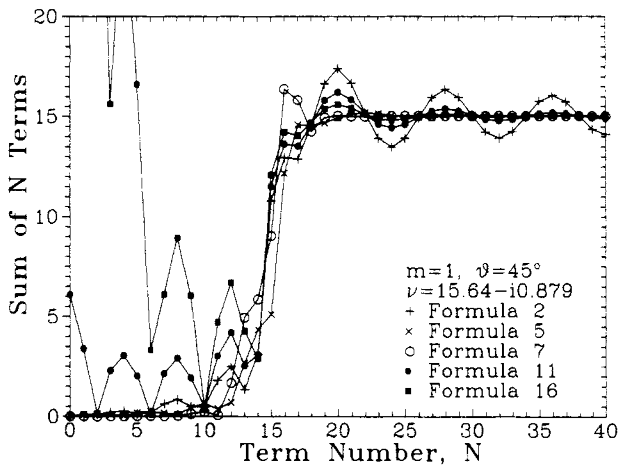

Figure 4. Sum to N terms of the formulae for $|P_\nu^1(\cos\vartheta)|$ with $\vartheta = 45°$ and $\nu = 15.64 - i0.879$. ($|P_\nu^1| = 15.028$.)

computer (using single precision variables). In our application, the two Legendre functions $P_\nu^0(-\cos\vartheta)$ and $P_\nu^1(-\cos\vartheta)$ have to be computed within a program iteration loop for fitting experimental data to a theoretical model so computation time is of importance. The simplicity of the current formulae (with the consequent ease of programming) and their physical significance (discussed in section 2 and section 5) are the major advantages of the work presented here.

7. Summary

A number of formulae have been derived which enable the Legendre functions $P_\nu^m(x)$ with ν complex, x real with $-1 < x \leqslant +1$ and $m = 0, 1$, to be computed efficiently for values of ν and ϑ for which use of the asymptotic expansion is inappropriate because

$|\nu|$ or ϑ is too small. The formulae all involve series expansions in terms of the zonal harmonic functions $P_n^m(x)$ which can be calculated rapidly using a recurrence relation. The formulae given in this paper are suitable for implementation on microcomputers and have all been tested numerically and shown to be valid.

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